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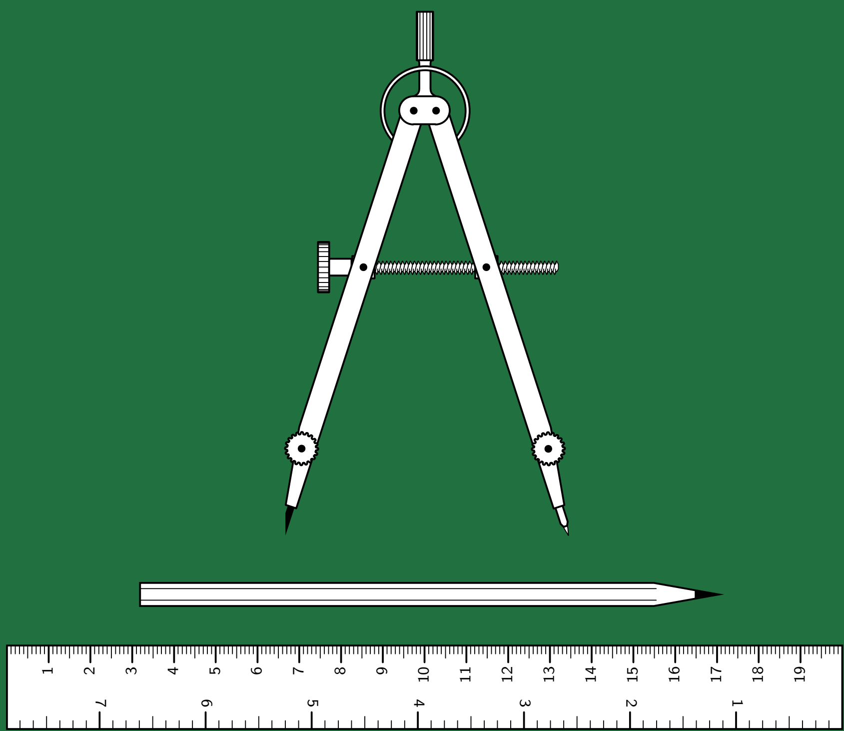
STEFAN LINDSTRÖM

LECTURES ON ENGINEERING MECHANICS

STATICS AND DYNAMICS



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Lectures on Engineering Mechanics: Statics and Dynamics
Stefan Lindström

Color print version

ISBN 978-91-981287-4-1
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Translated by Eva-Karin Lindström

Other versions of this book, with exercises:
ISBN 978-91-981287-5-8: SI version, Amazon, 2021
ISBN 978-91-981287-8-9: USC version, Amazon, 2021

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Preface

This textbook gives a concise description of elementary Engineering Mechanics including definitions and theorems. It is suitable for Bachelor's level engineering studies.

It is presumed that the reader is familiar with basic geometry (Appendix A.1), geometric vectors (Appendix A.2), linear systems of equations, ordinary differential equations including differential notation (Appendix A.3), and integrals in several dimensions (Appendix A.4). Besides this, the reader should be acquainted with the concepts of quantity, unit and dimension, and be able to determine whether a physical expression is dimensionally correct (Appendix B).

Acknowledgments

The author wishes to acknowledge his debt to Dr. Peter Schmidt, Dr. Lars Johansson and Dr. Ulf Edlund for their useful feedback.

Dr. Stefan Lindström
Linköping
June 2019

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Part I
Statics

1 Introduction

This chapter introduces some fundamental concepts of Mechanics, and outlines the field of Statics. It is necessary to be familiar with vectors and their properties (Appendix A.2) to grasp the key concepts, and to solve applied problems.

1.1 Fundamental concepts

Bodies and rigid bodies

A *body* occupies a finite region in space, and therefore, it has a volume. A body also has mass, and this mass is assumed to be continuously distributed within the region of the body.

All physical bodies can deform (change their shape). That is, the distance between the material points within the body can change. In some situations this deformation is very small and can be neglected. In such cases, we assume that the shape of the body is unchanging, and we call this a *rigid body model*.

Definition 1.1 (Rigid body)**.** A *rigid body* is a body with the constraint, that the distance between each pair of its material points cannot change.

Particles

A *particle* is a hypothetical object with mass, but without volume. Therefore, all its mass is concentrated to one point. In applications, we sometimes use a *particle model* for a body when its rotation and deformation do not affect the analysis to any great extent. Particularly, we can formulate the following *postulate*[1]:

Postulate 1.2 (Particle)**.** A body, or a part of a body, whose extension is sufficiently small to be neglected in a given situation, can be considered as a particle.[2]

[1] *Postulate* – an unproven statement with experimental support.

[2] J. B. Griffiths. *The theory of classical mechanics*. Cambridge University Press, 1985. ISBN 0-521-23760-2

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Position, velocity and acceleration

The position of a point, or a particle, in space is represented by its *position vector*[3]. We define the position vector of a point $\mathcal{P}$ as $\bar{r} \equiv \overline{\mathcal{OP}}$, where $\mathcal{O}$ denotes the origin of a coordinate system, *e.g.* a rectangular coordinate system with coordinates x, y and z, and corresponding basis vectors $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_z$. If the position of $\mathcal{P}$ changes with time t, the position vector becomes a vector-valued function (see Appendix A.2)

$$\bar{r}(t) = x(t)\bar{e}_x + y(t)\bar{e}_y + z(t)\bar{e}_z, \tag{1.1}$$

which can be interpreted as a directed path of motion (Fig. 1.1a). The *velocity* of $\mathcal{P}$ is defined as

$$\bar{v}(t) \equiv \frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = \dot{x}\bar{e}_x + \dot{y}\bar{e}_y + \dot{z}\bar{e}_z, \tag{1.2}$$

and it is oriented in the tangent direction of the path of motion. A superposed dot over a scalar function denotes the time derivative of that function. The *acceleration* of $\mathcal{P}$ is defined as

$$\bar{a}(t) \equiv \frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\bar{r}}{\mathrm{d}t^2} = \ddot{x}\bar{e}_x + \ddot{y}\bar{e}_y + \ddot{z}\bar{e}_z. \tag{1.3}$$

Thus, it describes the rate of change of the velocity. Two superposed dots over a scalar function denote the second time derivative of that function.

When a point or particle moves at a constant velocity $\bar{v}$, it is said to describe *uniform motion*, and $\bar{r}(t)$ becomes a rectilinear path (Fig. 1.1b). As a special case, when $\bar{v} = \bar{0}$, $\bar{r}(t)$ identifies a fixed point. In both cases, it follows from Eq. (1.3) that $\bar{a} = \bar{0}$.

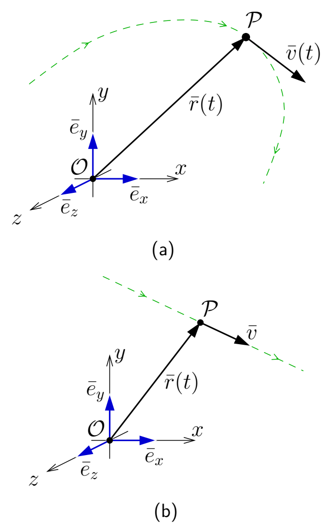

Figure 1.1: The path of motion $\bar{r}(t)$ of a point $\mathcal{P}$, with (a) a varying velocity $\bar{v}(t)$, or with (b) a constant velocity $\bar{v}$, and acceleration $\bar{a} = \bar{0}$.

Forces

When we place two objects sufficiently close to each other, or if they come into contact, the objects can affect each other's motion. If, for instance, one places a magnet near a steel pin, this pin will accelerate towards the magnet. The ability of two bodies to affect each other's motion is called *interaction*[4].

The concept of *force* is introduced to quantify the magnitude and direction of the interactions of an object with its surroundings. An interaction creates a force that makes the object accelerate. Otherwise, the object would remain still, or move uniformly in rectilinear motion. The force is defined through the laws of particle motion, formulated by Sir Isaac Newton in *Principia* (1687).

[3] Also called *radius vector*.

[4] Also called *reciprocal action*.

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1.2 Newton's laws of motion

Sir Isaac Newton postulated the following three laws of motion for particles:

1. *Law of inertia* A particle remains at rest, or moves in a straight line at a constant velocity, as long as the particle does not interact with any other object.

2. *Law of force and acceleration* For a particle with constant mass m, it holds that

$$\Sigma \bar{F} = m\bar{a}, \tag{1.4}$$

   where $\Sigma \bar{F}$ is the vector sum of all forces acting on the particle, and $\bar{a}$ is the acceleration of the particle.

3. *Law of action and reaction* If a particle exerts a force on another particle, then the latter exerts a force of equal magnitude, but opposite direction, on the former particle.

These laws are treated comprehensively in Chapter 8.

Inertial system

In order to describe motion, a coordinate system first needs to be specified. Newton's laws are only valid for particular coordinate systems called *inertial systems*. If a coordinate system is chosen so that the Law of inertia is valid, then the Law of force and acceleration and the Law of action and reaction become valid in that coordinate system as well. In coordinate systems that rotate or otherwise accelerate relative to an inertial system, Newton's three laws are not valid (Fig. 1.2).

Statics concerns mechanical systems for which all material points describe uniform motion with the same constant velocity in an inertial system.

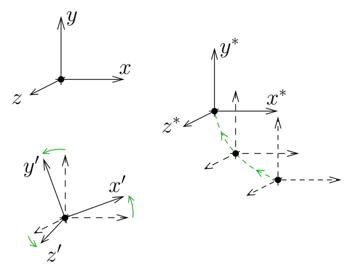

Figure 1.2: Given an inertial system xyz, where the Law of inertia is valid, any coordinate system $x'y'z'$ that rotates relative to this inertial system is not an inertial system. Coordinate systems $x^*y^*z^*$ for which the origin accelerates relative to the inertial system are not inertial systems either.

1.3 Forces in Classical Mechanics

Forces may act on a body if this body is in physical contact with another body. Forces can also act over a distance, *e.g.*, through a gravitational or magnetic field. We measure force in the SI unit of newton (N), or in the USC unit of pound-force ($\mathrm{lb_f}$), where

$$1\,\mathrm{N} = 1\,\frac{\mathrm{kg{\cdot}m}}{\mathrm{s}^2}, \qquad 1\,\mathrm{lb_f} \approx 4.4482\,\mathrm{N}.$$

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Universal law of gravitation

According to *Newton's universal law of gravitation*, every pair of particles affect each other with gravitational forces. The gravitational force is a central force of attraction. That is, a pair of particles are drawn towards each other, and the gravitational force acts along a straight line connecting these particles (Fig. 1.3).

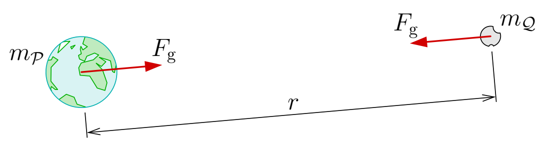

Figure 1.3: Newton's universal law of gravitation for particles applied to the Earth's interaction with the Moon.

Postulate 1.3 (Gravitational force)**.** Between two particles with masses $m_\mathcal{P}$ and $m_\mathcal{Q}$, respectively, there is an attractive force with magnitude

$$F_g = G_g \frac{m_\mathcal{P} m_\mathcal{Q}}{r^2}, \tag{1.5}$$

where

$$G_g = 6.67408 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg{\cdot}s}^2} \approx 3.4397 \cdot 10^{-8} \frac{\mathrm{ft}^4}{\mathrm{lb_f{\cdot}s}^4} \tag{1.6}$$

is the *gravitational constant*, and r is the distance between the particles.

One consequence of the Universal law of gravitation is that a body with mass m near the surface of the Earth is affected by a gravitational force directed towards the center of the Earth. This gravitational force is distributed over the region occupied by the body. However, in many applications gravitation can be modeled as *one* force acting in one point. This force is the *weight* of the body, and has the magnitude mg, where g is the *local gravity constant*[5]. The value of the local gravity constant varies around the world. The *standard gravity*[6]

$$g_n \equiv 9.80665 \frac{\mathrm{N}}{\mathrm{kg}} \approx 32.174 \frac{\mathrm{ft}}{\mathrm{s}^2}$$

is often used in problem solving.

[5] Also called the *acceleration of gravity*.

[6] Bureau International des Poids et Mesures. The International System of Units (SI), 2006

Contact forces

Two bodies in physical contact with each other interact through *contact forces*. These contact forces are distributed over the touching surfaces of the respective bodies. One example is the force that appears when you press your hand against a wall (Fig. 1.4ab). Your hand exerts a pressure on the wall, which can be represented by a force $\bar{F}$. Conversely, the wall exerts a force $-\bar{F}$ on your hand according to the Law of action and reaction.

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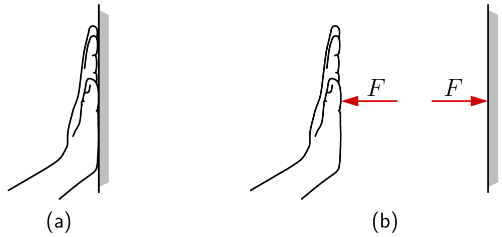

Figure 1.4: (a) Your hand presses against a wall. (b) Your hand and the wall are subjected to equally large but oppositely directed contact forces.

Spring forces

Spring forces appear when a spring is elongated or compressed. When a spring is not affected by any force, it assumes its *natural length* ℓ_0 (Fig. 1.5a). If oppositely directed forces of equal magnitude F_s act in both ends of this spring, it will change its length to ℓ (Fig. 1.5b). For a *linear spring*, the relation

$$F_s = k(\ell - \ell_0) \tag{1.7}$$

holds, where k is the *spring constant* with the SI units of N/m and the USC units of lb_f/ft.

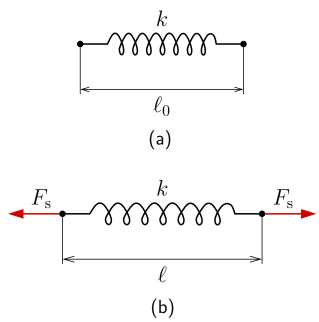

Figure 1.5: (a) Unloaded spring with its natural length. (b) The same spring when elongated by a tensile force.

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2

Force-couple systems

2.1 Forces

A body interacts with its surroundings through *forces*. These can be *body forces* that act over the region in space occupied by the body. Gravitational and electromagnetic forces are examples of body forces. A body can also be affected by *contact forces* that are distributed across the surface of the body (Fig. 2.1). For rigid bodies, the body and contact forces can be represented by concentrated forces that act in distinct points on the rigid body.

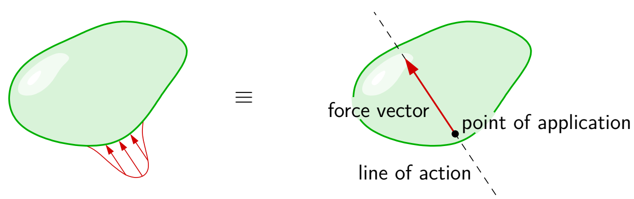

Figure 2.1: A contact force is distributed across part of the surface of a rigid body. It is modeled by a force vector that acts in a point of application on the rigid body.

Postulate 2.1. A *force* is a vector quantity $\bar{F}$, which is assigned a *point of application* $\mathcal{P}$.

The effect of a force on a body is determined by the magnitude and the direction of the force, and by its point of application. The force vector and the point of application define a line called the *line of action* (Fig. 2.1).

As is the case for all vectors, the force vector can be written as a sum of its components (Fig. 2.2),

$$\bar{F} = F_x\bar{e}_x + F_y\bar{e}_y + F_z\bar{e}_z, \tag{2.1}$$

or as a scalar F multiplied by a unit vector:

$$\bar{F} = F\bar{e}_F. \tag{2.2}$$

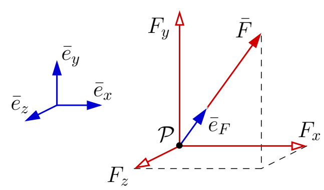

Figure 2.2: A force $\bar{F}$ acting in its point of application $\mathcal{P}$. Arrows with open arrowheads represent components of the force.

In Eq. (2.2), F is allowed to be negative, so that $F = \pm|\bar{F}|$. The force component of $\bar{F}$ in the λ direction is

$$F_\lambda = \bar{F} \cdot \bar{e}_\lambda = |\bar{F}| \cos\varphi, \tag{2.3}$$

where φ is the angle between $\bar{F}$ and $\bar{e}_\lambda$ (Fig. 2.3).

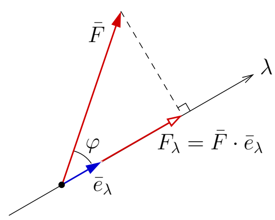

Figure 2.3: The force component of $\bar{F}$ w.r.t. the λ direction.

2.2 Moments and couples

Moments of force

To create a turning action about an axis, for instance when turning a bolt around its longitudinal axis, one applies a force in a point at some distance away from this axis (Fig. 2.4). The turning action of the force is called the *moment* of the force.

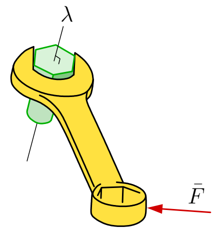

Figure 2.4: A force with its point of application at a distance from an axis λ creates a turning action about this axis.

Definition 2.2 (Moment of force)**.** Let $\bar{F}$ be a force with point of application $\mathcal{P}$. Then, the *moment* of the force $\bar{F}$, w.r.t. an arbitrary point $\mathcal{A}$, is the vector

$$\bar{M}_\mathcal{A} \equiv \overline{\mathcal{AP}} \times \bar{F}, \tag{2.4}$$

where $\mathcal{A}$ is the *moment reference point.*

Recollecting Def. A.18 of the cross product, the direction of the moment vector $\bar{M}_\mathcal{A}$ is given by the right-hand rule (Fig. 2.5). Therefore, the moment of force is perpendicular to the plane spanned by $\overline{\mathcal{AP}}$ and $\bar{F}$. The magnitude of $\bar{M}_\mathcal{A}$ is

$$\begin{aligned} |\bar{M}_\mathcal{A}| &= |\overline{\mathcal{AP}} \times \bar{F}| = \{\text{Eq. (A.19)}\} \\ &= |\overline{\mathcal{AP}}||\bar{F}| \sin\varphi \\ &= |\bar{F}| d_\perp, \end{aligned} \tag{2.5}$$

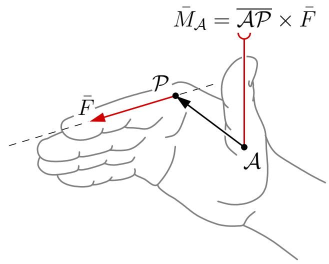

Figure 2.5: The right-hand rule for the moment of force. Keep the palm of your right hand aligned with the lever arm and angle your fingers in the direction of the force. Then your thumb will point in the direction of the moment vector.

where $d_\perp = |\overline{\mathcal{AP}}| \sin\varphi$ is called the *lever arm* or the *moment arm*, and φ is the angle between $\overline{\mathcal{AP}}$ and $\bar{F}$ (Fig. 2.6). The moment vector is depicted as an arrow with a U-shaped head. The moment of force, w.r.t. an axis λ with direction vector $\bar{e}_\lambda$, is defined as

$$M_{\mathcal{A}\lambda} \equiv \bar{M}_\mathcal{A} \cdot \bar{e}_\lambda, \tag{2.6}$$

where $\mathcal{A}$ is an arbitrary point on the λ axis.

Theorem 2.3. Let n forces, $\bar{F}_1, \ldots, \bar{F}_n$, act in the same point $\mathcal{P}$. The sum of their moments of force, w.r.t. an arbitrary point $\mathcal{A}$, is then equal to the moment of force of the sum of the force vectors, w.r.t. $\mathcal{A}$:

$$\sum_{i=1}^{n} \overline{\mathcal{AP}} \times \bar{F}_i = \overline{\mathcal{AP}} \times \sum_{i=1}^{n} \bar{F}_i. \tag{2.7}$$

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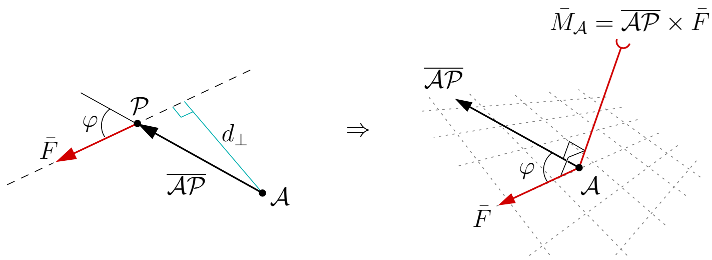

Figure 2.6: A force with force vector $\bar{F}$ and point of application $\mathcal{P}$ gives a moment of force $\bar{M}_\mathcal{A}$ w.r.t. $\mathcal{A}$, that is perpendicular to the plane spanned by $\overline{\mathcal{AP}}$ and $\bar{F}$.

Proof. The moment of force of the sum of the force vectors, w.r.t. $\mathcal{A}$, is

$$\begin{aligned}
\overline{\mathcal{AP}} \times \sum_{i=1}^{n} \bar{F}_i &= \overline{\mathcal{AP}} \times (\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n) = \{\text{Eq. (A.21b)}\} \\
&= \overline{\mathcal{AP}} \times \bar{F}_1 + \overline{\mathcal{AP}} \times (\bar{F}_2 + \cdots + \bar{F}_n) = \{\text{repeat (A.21b)}\} \\
&= \overline{\mathcal{AP}} \times \bar{F}_1 + \overline{\mathcal{AP}} \times \bar{F}_2 + \cdots + \overline{\mathcal{AP}} \times \bar{F}_n \\
&= \sum_{i=1}^{n} \overline{\mathcal{AP}} \times \bar{F}_i. \qquad \square
\end{aligned}$$

When analyzing Statics problems, it is often convenient to decompose the forces into their vector components (Fig. 2.7). According to Theorem 2.3, the moment of a force is then given by the sum of the moments of force of its components.

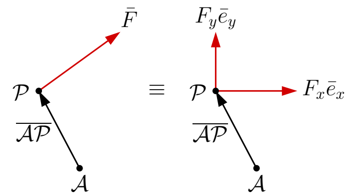

Figure 2.7: The moment of a force equals the sum of the moments of force of its vector components: $\overline{\mathcal{AP}} \times \bar{F} = \overline{\mathcal{AP}} \times F_x\bar{e}_x + \overline{\mathcal{AP}} \times F_y\bar{e}_y$ (2D).

Couples

Definition 2.4 (Force pair)**.** A *force pair* consists of two forces, $\bar{F}$ with point of application $\mathcal{P}$ and $-\bar{F}$ with point of application $\mathcal{Q}$ (Fig. 2.8).

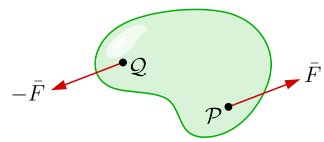

Figure 2.8: Illustration of a force pair.

A trivial yet important property of the force pair is that the sum of its forces is $\bar{F} + (-\bar{F}) = \bar{0}$. Consequently, the only effect of a force pair is a turning action.

Definition 2.5 (Couple)**.** A *couple* $\bar{C}$ is the sum of the moments of force from a force pair w.r.t. an arbitrary point $\mathcal{A}$.

Theorem 2.6. For an arbitrary force pair, $\bar{F}$ with point of application $\mathcal{P}$ and $-\bar{F}$ with point of application $\mathcal{Q}$ (Fig. 2.9), its couple is

$$\bar{C} = \overline{\mathcal{QP}} \times \bar{F}. \tag{2.8}$$

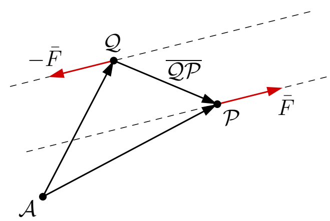

Figure 2.9: A force pair creating a couple $\bar{C} = \overline{\mathcal{QP}} \times \bar{F}$.

Proof. From Def. 2.5, it follows that the couple of a force pair, w.r.t. an arbitrary point $\mathcal{A}$, is

$$\begin{aligned}
\bar{C} &= \overline{\mathcal{AP}} \times \bar{F} + \overline{\mathcal{AQ}} \times (-\bar{F}) \\
&= \overline{\mathcal{AP}} \times \bar{F} - \overline{\mathcal{AQ}} \times \bar{F} = \{\text{Eq. (A.21b)}\}
\end{aligned}$$

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$$= (\overline{\mathcal{AP}} - \overline{\mathcal{AQ}}) \times \bar{F}$$
$$= (\overline{\mathcal{QA}} + \overline{\mathcal{AP}}) \times \bar{F} = \{\text{Parallelogram law}\}$$
$$= \overline{\mathcal{QP}} \times \bar{F}. \qquad \square$$

An example of a force pair is the action of a screwdriver on a slotted screw (Fig. 2.8). There are two contact points, $\mathcal{P}$ and $\mathcal{Q}$, between the screw head and the tip of the screwdriver, where two oppositely directed forces with the same magnitude act on the screw. The couple is independent of the choice of moment reference point. Therefore, this couple is a free vector that can be translated to an arbitrary point (Fig. 2.10).

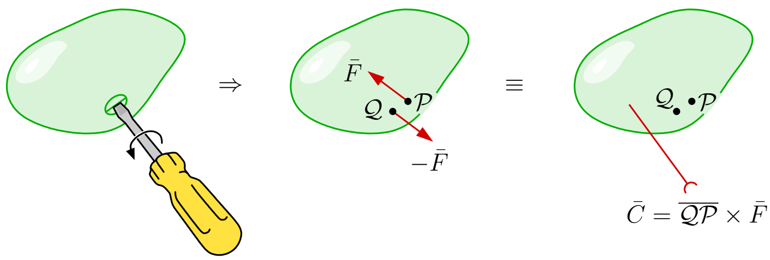

Figure 2.10: A screwdriver creates a turning action owing to the force pair acting on the slotted screw. The couple can be regarded as a free vector. That is, it does not act in any specific point on the rigid body.

2.3 Force-couple systems

Several forces and couples that act simultaneously on a rigid body form a forces-couple system.

Definition 2.7 (Force-couple system)**.** A *force-couple system* Γ is a set of $n \geq 0$ forces $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n$ with points of applications $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, and a set of $m \geq 0$ couples $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_m$ (Fig. 2.11).

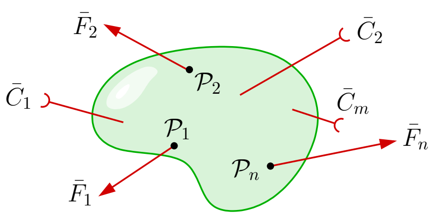

Figure 2.11: A force-couple system Γ with a number of forces and couples acting on a rigid body.

Force sums and moment sums

Definition 2.8 (Force sum)**.** For a force-couple system, with notation as in Def. 2.7, the *force sum* is the vector

$$\Sigma\bar{F} \equiv \sum_{i=1}^{n} \bar{F}_i. \tag{2.9}$$

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Observe that the force sum is not assigned any point of application. Therefore, it does not fulfill the conditions in Postulate 2.1 for being a proper force.

Definition 2.9 (Moment sum)**.** For a force-couple system Γ, with notation as in Def. 2.7, the *moment sum* w.r.t. an arbitrary point $\mathcal{A}$ is the vector

$$\Sigma\bar{M}_{\mathcal{A}} \equiv \sum_{i=1}^{n} \overline{\mathcal{A}\mathcal{P}}_i \times \bar{F}_i + \sum_{j=1}^{m} \bar{C}_j. \tag{2.10}$$

Thus, one obtains the moment sum of a force-couple system, w.r.t. a point $\mathcal{A}$, by summing all the moments of force of the system w.r.t. $\mathcal{A}$, and all the couples of the system.

Theorem 2.10 (Transfer theorem of the moment sum)**.** For a force-couple system, with notation as in Def. 2.7, and for two arbitrary points $\mathcal{A}$ and $\mathcal{B}$, it holds that

$$\Sigma\bar{M}_{\mathcal{B}} = \Sigma\bar{M}_{\mathcal{A}} + \overline{\mathcal{B}\mathcal{A}} \times \Sigma\bar{F}, \tag{2.11}$$

where $\Sigma\bar{M}_{\mathcal{A}}$ and $\Sigma\bar{M}_{\mathcal{B}}$ are moment sums w.r.t. $\mathcal{A}$ and $\mathcal{B}$, respectively, and where $\Sigma\bar{F}$ is the force sum of the system.

Proof. Definition 2.9 gives

$$\begin{aligned}
\Sigma\bar{M}_{\mathcal{B}} &= \sum_{i=1}^{n} \overline{\mathcal{B}\mathcal{P}}_i \times \bar{F}_i + \sum_{j=1}^{m} \bar{C}_j = \{\text{Parallelogram law}\} \\
&= \sum_{i=1}^{n} \left(\overline{\mathcal{B}\mathcal{A}} + \overline{\mathcal{A}\mathcal{P}}_i\right) \times \bar{F}_i + \sum_{j=1}^{m} \bar{C}_j = \{\text{Eq. (A.21b)}\} \\
&= \sum_{i=1}^{n} \overline{\mathcal{B}\mathcal{A}} \times \bar{F}_i + \underbrace{\sum_{i=1}^{n} \overline{\mathcal{A}\mathcal{P}}_i \times \bar{F}_i + \sum_{j=1}^{m} \bar{C}_j}_{=\Sigma\bar{M}_{\mathcal{A}}} = \{\text{Theorem 2.3}\} \\
&= \overline{\mathcal{B}\mathcal{A}} \times \Sigma\bar{F} + \Sigma\bar{M}_{\mathcal{A}}.
\end{aligned}$$

□

Reduced force-couple systems

Definition 2.11 (Reduced force-couple system)**.** A *reduced force-couple system* $\Gamma_{\mathcal{A}}$ of a force-couple system Γ, w.r.t. a *point of reduction* $\mathcal{A}$, consists of the force sum $\Sigma\bar{F}$ of Γ acting in $\mathcal{A}$, and of a couple $\Sigma\bar{M}_{\mathcal{A}}$, being the moment sum of Γ w.r.t. $\mathcal{A}$ (Fig. 2.12).

The reduced force-couple system $\Gamma_{\mathcal{A}}$ is equivalent to Γ in the sense that Γ and $\Gamma_{\mathcal{A}}$ would induce the same motion for a rigid body.

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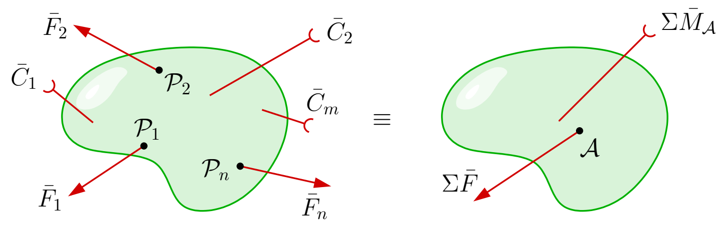

Figure 2.12: A force-couple system Γ, with an arbitrary number of forces and couples, is equivalent to its reduced force-couple system $\Gamma_\mathcal{A}$, w.r.t. an arbitrary point $\mathcal{A}$.

Definition 2.12 (Zero system)**.** If a force-couple system has the force sum $\Sigma\bar{F} = \bar{0}$ and the moment sum $\Sigma\bar{M}_\mathcal{A} = \bar{0}$, w.r.t. some point $\mathcal{A}$, then the force-couple system is a *zero system*.

Theorem 2.13. If a force-couple system is a zero system, then its moment sum is $\Sigma\bar{M}_\mathcal{B} = \bar{0}$ for every point $\mathcal{B}$.

Proof. When a force-couple system, with notation as in Def. 2.7, is a zero system, we have $\Sigma\bar{F} = \bar{0}$ and $\Sigma\bar{M}_\mathcal{A} = \bar{0}$ for some point $\mathcal{A}$. Consequently, by Theorem 2.10, it holds that

$$\begin{aligned}\Sigma\bar{M}_\mathcal{B} &= \Sigma\bar{M}_\mathcal{A} + \overline{\mathcal{BA}} \times \Sigma\bar{F}\\ &= \bar{0} + \overline{\mathcal{BA}} \times \bar{0}\\ &= \bar{0}.\end{aligned}$$ □

Theorem 2.13 states that a zero system is always a zero system, regardless of the choice of moment reference point.

2.4 Planar force-couple systems

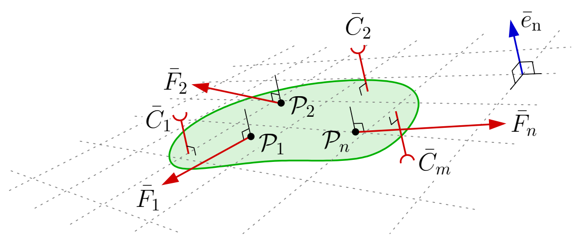

Figure 2.13: A planar force-couple system w.r.t. a reference plane with unit normal $\bar{e}_\mathrm{n}$.

Definition 2.14 (Planar force-couple system)**.** A force-couple system, with notation as in Def. 2.7, is *planar* if there exists a plane called the *reference plane*, such that each point of application $\mathcal{P}_i$ is located in this reference plane, and such that

$$\begin{aligned}\bar{F}_i &\perp \bar{e}_\mathrm{n}, \quad i = 1, \ldots, n,\\ \bar{C}_j &\parallel \bar{e}_\mathrm{n}, \quad j = 1, \ldots, m,\end{aligned}$$

where $\bar{e}_\mathrm{n}$ is the unit normal of the reference plane (Fig. 2.13).

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For a planar force-couple system and a moment reference point $\mathcal{A}$ located in its reference plane, the moments of force and the couples are oriented in the $\pm\bar{e}_\mathrm{n}$ direction. Therefore, each moment of force and each couple can be uniquely represented by its normal component, which is a scalar. Figure 2.14 depicts a planar force-couple system with the xy plane as the reference plane. A scalar representation is used for the moments, which is indicated by a curved arrow for each couple $C_1, \ldots, C_m$, corresponding to the $\bar{e}_z$ or $-\bar{e}_z$ directions according to the right-hand rule (Fig. 2.15).

Let $\bar{F}$ denote a force, with point of application $\mathcal{P}$, belonging to a planar force-couple system (Fig. 2.16). The moment $\bar{M}_\mathcal{A} = \overline{\mathcal{AP}} \times \bar{F}$ of this force can be written as $\bar{M}_\mathcal{A} = M_\mathcal{A}\bar{e}_\mathrm{n}$, where

$$
\begin{aligned}
M_\mathcal{A} &= \pm|\bar{M}_\mathcal{A}| = \{\text{Def. 2.2}\} \\
&= \pm|\overline{\mathcal{AP}} \times \bar{F}| = \{\text{Eq. (A.19)}\} \\
&= \pm|\overline{\mathcal{AP}}||\bar{F}|\sin\varphi.
\end{aligned}
$$

Here, φ is the angle between $\overline{\mathcal{AP}}$ and $\bar{F}$. Since the distance from $\mathcal{A}$ to the line of action of the force is $d_\perp = |\overline{\mathcal{AP}}|\sin\varphi$, it follows that

$$
M_\mathcal{A} = \pm F d_\perp. \tag{2.12}
$$

As pointed out above, the direction of the moment of force is dictated by the right-hand rule. The counterclockwise turning of the moment of force in Fig. 2.16 is oriented in the $\bar{e}_z$ direction. If we select the normal of the reference plane as $\bar{e}_\mathrm{n} = \bar{e}_z$, the scalar representation $M_\mathcal{A}$ of this moment of force will be positive. Clockwise oriented moments of force, on the other hand, become negative. The converse applies if we select $\bar{e}_\mathrm{n} = -\bar{e}_z$.

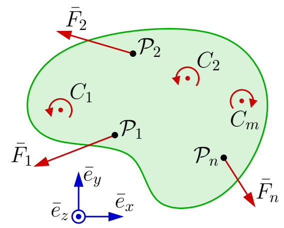

Figure 2.14: A planar force-couple system with the xy plane as its reference plane. The moments and couples of the system can be represented by scalars.

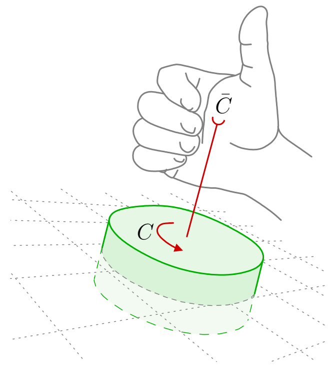

Figure 2.15: The vector direction of a couple represented by a scalar C and a curved arrow is identified by aligning the fingers of the right hand, except the thumb, with the curved arrow. The thumb then points in the vector direction of the couple.

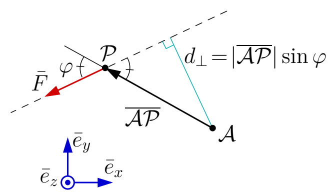

Figure 2.16: Geometry for the moment of the force F in a planar force-couple system, with the xy plane as its reference plane. The lever arm is denoted by $d_\perp$.

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3
Static equilibrium

3.1 Equilibrium equations

Definition 3.1 (Static equilibrium)**.** A body is in *static equilibrium* if every material point in the body has the same constant velocity relative to an inertial system.

Since Def. 3.1 requires that the velocities of the material points are equal and constant, it follows that all those points move along straight, parallel paths. Such motion is called *rectilinear translation* (Fig. 3.1). A rigid body is said to be at *rest*, if it is in static equilibrium and the velocity of the material points is zero in the chosen inertial system.

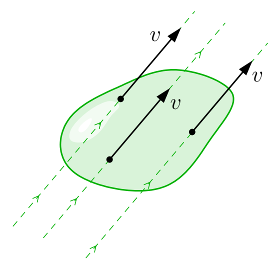

Figure 3.1: Static equilibrium implies that a rigid body describes rectilinear translation. Every point in the body moves with the same constant velocity.

Static equilibrium is defined from the motion of the body, not from the forces that act on the body. If a rigid body is in static equilibrium, we need a postulate to identify force-couple systems that maintains this equilibrium:

Postulate 3.2 (Equilibrium conditions)**.** A rigid body in static equilibrium remains in static equilibrium if the force-couple system that acts on this rigid body is a zero system,

$$\Sigma\bar{F} = \bar{0}, \tag{3.1a}$$

$$\Sigma\bar{M}_{\mathcal{A}} = \bar{0}, \tag{3.1b}$$

where $\Sigma\bar{F}$ is the force sum, and $\Sigma\bar{M}_{\mathcal{A}}$ is the moment sum of the force-couple system w.r.t. an arbitrary point $\mathcal{A}$.

Equation (3.1a) is called the *Force equilibrium equation* and Eq. (3.1b) is called the *Moment equilibrium equation*. According to Theorem 2.13, the moment reference point of the Moment equilibrium equation can be freely selected.

The Force and the Moment equilibrium equations are vector equations. According to Eq. (A.11) they can be written on component form

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as a system of six scalar equations:

$$\begin{cases} \Sigma F_x = 0, & \Sigma M_{\mathcal{A}x} = 0, \\ \Sigma F_y = 0, & \Sigma M_{\mathcal{A}y} = 0, \\ \Sigma F_z = 0, & \Sigma M_{\mathcal{A}z} = 0. \end{cases}$$

Equilibrium of planar force-couple systems

For a planar force-couple system, the Force and the Moment equilibrium equations can be simplified by choosing a coordinate system with two coordinate axes in the reference plane. If the xy plane is placed in the reference plane (Fig. 2.14), so that $\bar{e}_{\mathrm{n}} = \bar{e}_z$, then Def. 2.14 gives:

$$\bar{F}_i \perp \bar{e}_z \quad \Leftrightarrow \quad F_{iz} = 0, \qquad i = 1, 2, \ldots \qquad \Rightarrow$$
$$\Sigma F_z = 0.$$

Moreover, all the moments of force and all the couples are oriented in the z direction (Sect. 2.4), so that

$$\Sigma M_{\mathcal{A}x} = \Sigma M_{\mathcal{A}y} = 0,$$

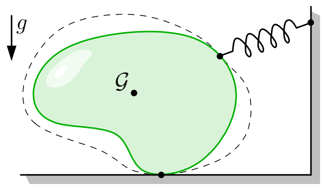
where $\mathcal{A}$ is a moment reference point in the reference plane. In conclusion, only three nontrivial, scalar equilibrium equations remain for the planar force-couple system:

$$\begin{cases} \Sigma F_x = 0, \\ \Sigma F_y = 0, \\ \Sigma M_{\mathcal{A}z} = 0. \end{cases}$$

3.2 Free-body diagrams

A *free-body diagram* is an aid to identify all forces and couples that act on a mechanical system. When drawing a free-body diagram, the body is isolated from its surroundings. That is, the surrounding objects are removed and instead their action on the body is represented by forces and couples. When drawing a free-body diagram, one observes the following protocol:

1. Decide which body to analyze. Here, a dashed line circumscribes the selected body.



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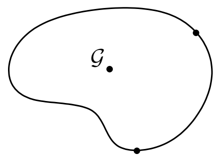
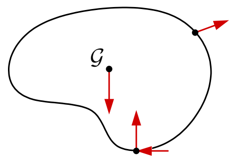
2. Draw a diagram containing *only* the selected body.



3. Represent the action of the surrounding objects on the body with forces and couples.



The surroundings of the analyzed body may exert forces through force fields, *e.g.* the force of gravity, as well as contact forces that arise at points of physical contact between the body and the surrounding objects.

The force of gravity

The action of *gravity* on a rigid body near the surface of the Earth is modeled as a force: the *force of gravity* acting in the center of gravity $\mathcal{G}$ of a body (Fig. 3.2). The force of gravity is directed towards the center of the Earth, and it has the magnitude mg, where m is the mass of the body, and g is the local gravity constant. Gravity will be further studied in Chapter 4.

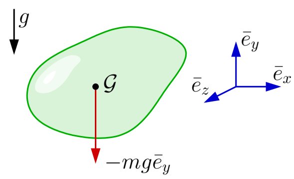


Figure 3.2: Gravity acting on a rigid body near the surface of the Earth. The force of gravity has the magnitude mg, acting in the center of gravity $\mathcal{G}$ of the rigid body.

Constraint forces and couples

If a rigid body is in physical contact with surrounding objects, and therefore is prevented from moving or rotating freely, then *constraint forces* and *constraint couples* arise at the points of contact.

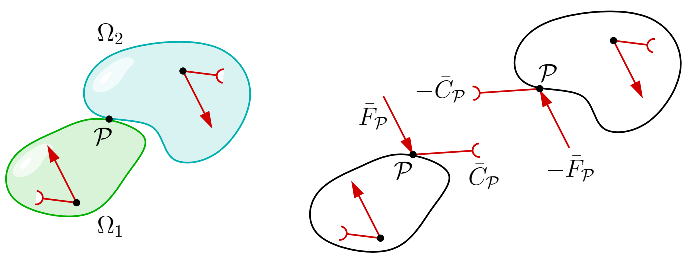


Figure 3.3: Two bodies, Ω_1 and Ω_2, with a point contact $\mathcal{P}$. The free-body diagrams show contact forces and couples between the bodies.

Consider a *point contact* between two bodies, Ω_1 and Ω_2, that are in physical contact with each other at a common point $\mathcal{P}$. Generally, this sort of contact creates a couple $\bar{C}_{\mathcal{P}}$ and a force $\bar{F}_{\mathcal{P}}$, which act in $\mathcal{P}$ on Ω_1. According to an extended version of the Law of action and reaction, the contact also creates a couple $-\bar{C}_{\mathcal{P}}$ and a force $-\bar{F}_{\mathcal{P}}$, which act in $\mathcal{P}$ on Ω_2 (Fig. 3.3).

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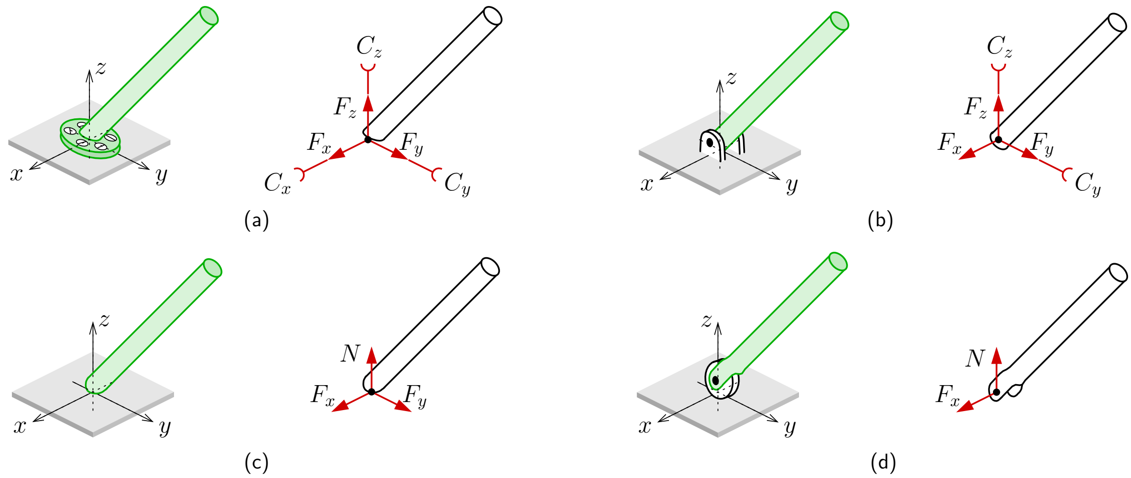

Figure 3.4: Free-body diagrams for different supports. (a) Rigid supports, for instance welds, bolted joints and glued joints, where forces and couples can arise in any direction. (b) A hinge where a pin allows rotation around the x axis. As a consequence, $C_x = 0$. (c) At a friction contact of a rounded body, rotations are allowed through rolling against the surface: $C_x = C_y = 0$. By neglecting friction for rotation around the normal axis, we obtain $C_z = 0$. (d) A wheel eliminates one of the friction force components, $F_y = 0$, while rotation is allowed around all axes: $C_x = C_y = C_z = 0$.

The point contact model is used for different types of supports and joints between bodies, such as welds, hinges, bearings and so on. The type of joint determines the direction of the constraint force and couple according to the two following principles:

1. If a joint at $\mathcal{P}$ allows for Ω_1 to translate freely relative to Ω_2 in a direction $\bar{e}_\lambda$, then

   $$\bar{F}_{\mathcal{P}} \cdot \bar{e}_\lambda = 0.$$

   An example is the y direction in Fig. 3.4d, where $F_y = 0$.

2. If a joint at $\mathcal{P}$ allows for Ω_1 to rotate freely relative to Ω_2 around an axis through $\mathcal{P}$ with direction vector $\bar{e}_\lambda$, then

   $$\bar{C}_{\mathcal{P}} \cdot \bar{e}_\lambda = 0.$$

   An example is the x direction in Fig. 3.4b where $C_x = 0$.

Thus, constraint forces can only arise in directions where relative displacement is restricted. Similarly, constraint couples can only arise in directions where relative rotation is restricted.

There are many different types of supports. Therefore, one needs to formulate a suitable point contact model for every new case. Some examples are given in Fig. 3.4. When a new type of support is encountered, first assume that all the components of the constraint force and couple are non-zero. From that outset, methodically proceed to eliminate components that lack constraint.

Strings and pulleys

A *string* is an idealized rope, wire or the like, which can be regarded as inextensible and massless. A stretched string is loaded only by a tensile

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force $T > 0$ in the longitudinal direction of the string. Thus, the ends of a segment of a stretched string are loaded by two forces $\bar{T}$ and $-\bar{T}$, respectively, parallel to the string (Fig. 3.5a).

When a string runs over a frictionless *pulley* with negligible mass, the tensile force will be the same in both ends of the string. This becomes clear from the moment equilibrium w.r.t. the hub of the pulley (Fig. 3.5b).

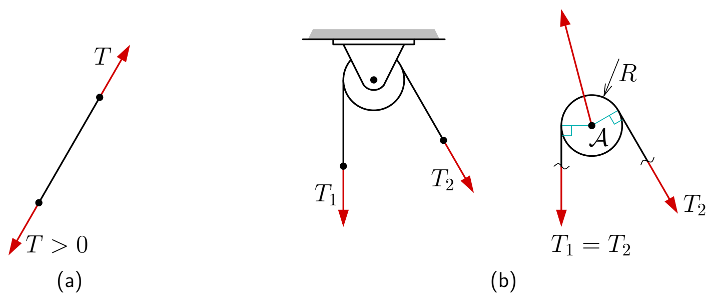

Figure 3.5: (a) A stretched string is loaded by two oppositely directed forces, which are parallel to the string. (b) A string runs over a frictionless pulley. A moment equilibrium for the pulley w.r.t. $\mathcal{A}$ gives $RT_1 - RT_2 = 0$, showing that $T_1 = T_2$.

Two-force members

An important special case of equilibrium is when precisely two forces act on a rigid body, which is then called a *two-force member*.

Theorem 3.3 (Two-force member)**.** If precisely two nonzero forces, and no couple, act on a rigid body in static equilibrium, then those forces have the same magnitude, are oppositely directed and have coincident lines of action (Fig. 3.6).

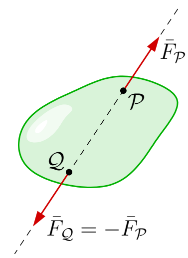

Figure 3.6: A two-force member in static equilibrium, where the lines of action of the forces are coincident.

Proof. Let two arbitrary forces, $\bar{F}_{\mathcal{P}}$ with point of application $\mathcal{P}$ and $\bar{F}_{\mathcal{Q}}$ with point of application $\mathcal{Q}$, act on a rigid body in static equilibrium. The force equilibrium gives

$$\bar{F}_{\mathcal{P}} + \bar{F}_{\mathcal{Q}} = \bar{0},$$

so that $\bar{F}_{\mathcal{P}} = -\bar{F}_{\mathcal{Q}}$, showing that the forces have the same magnitude, and that they are oppositely directed. Thus, their lines of action are parallel.

A moment equilibrium w.r.t. $\mathcal{P}$ gives (Fig. 3.7)

$$\begin{aligned} \overline{\mathcal{PQ}} \times \bar{F}_{\mathcal{Q}} = \bar{0} \quad &\Leftrightarrow \quad \{\text{Eq. (A.19)}\} \quad \Leftrightarrow \\ |\overline{\mathcal{PQ}}||\bar{F}_{\mathcal{Q}}| \sin\varphi = 0 \quad &\Leftrightarrow \quad \{\bar{F}_{\mathcal{Q}} \neq \bar{0}\} \quad \Leftrightarrow \\ |\overline{\mathcal{PQ}}| \sin\varphi = 0, & \end{aligned}$$

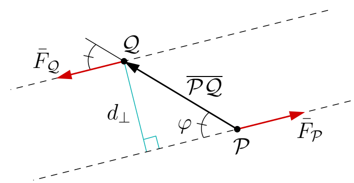

Figure 3.7: Geometry for the proof of Theorem 3.3.

where φ is the angle between $\overline{\mathcal{PQ}}$ and $\bar{F}_{\mathcal{Q}}$. The perpendicular distance between the lines of action is (Fig. 3.7)

$$d_\perp = |\overline{\mathcal{PQ}}| \sin\varphi = 0.$$

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Therefore, the lines of action must coincide. □

Massless bars attached to hinges are prototypical two-force members (Fig. 3.8). The analysis of multi-body mechanical systems can sometimes be simplified considerably by exploiting this property.

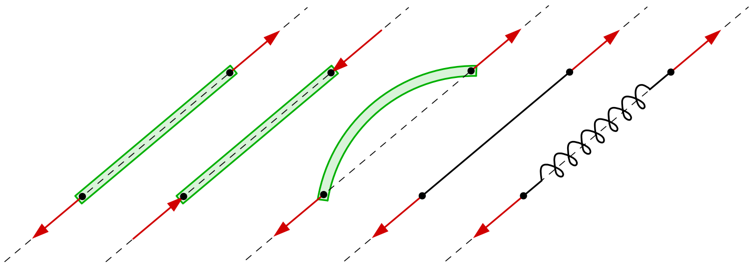

Figure 3.8: Bars, strings and springs with negligible mass, free from couples at their joints, are two-force members.

3.3 Multi-body mechanical systems

When a construction contains several different members, that are all in static equilibrium, the force-couple system of each member must be a zero system. It can be shown that it is a necessary condition for static equilibrium of a multi-body system, that a zero system of external forces and couples acts on the mechanical system.

When analyzing a multi-body system, it is permissible to draw the free-body diagram comprising several connected rigid bodies. For instance, consider the excavator in Fig. 3.9a. Depending on the problem formulation, it can be convenient to either draw the free-body diagram for the entire excavator (Fig. 3.9b), or to draw the free-body diagram for each of its members (Fig. 3.9c). The latter alternative is more suitable if the problem formulation concerns internal forces between the members of the construction.

The free-body diagrams of the members of the excavator (Fig. 3.9c) show some useful principles: Forces and reaction forces arise in the contact points between each pair of members. According to the Law of action and reaction, the force and the reaction force have the same magnitude, but opposite directions. The hydraulic cylinder is assumed to be massless, and therefore it can be regarded as a two-force member. Thus, the forces that act in its ends have the same magnitude, are oppositely directed, and their lines of action coincide (Theorem 3.3). The equilibrium equations can be formulated for each member, or for the entire mechanical system.

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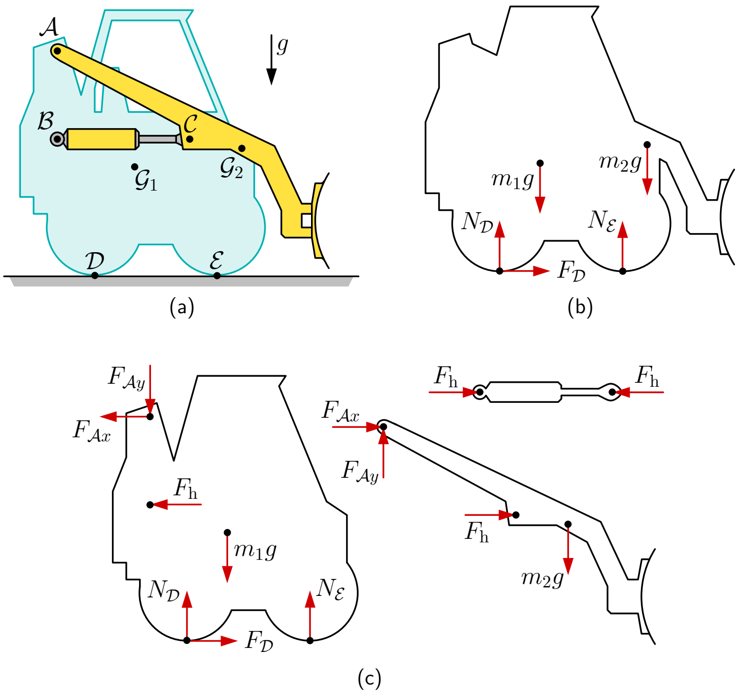

Figure 3.9: (a) An excavator consisting of a vehicle, with center of gravity $\mathcal{G}_1$ and mass m_1, a massless hydraulic cylinder $\mathcal{BC}$, and a scraper blade arm with center of gravity $\mathcal{G}_2$ and mass m_2. The front wheels are unbraked. (b) A free-body diagram of the whole construction. (c) Free-body diagrams of the members of the mechanical system, where the hydraulic cylinder is regarded as a two-force member.

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